

# Nonholonomic LL systems on central extensions and the hydrodynamic Chaplygin sleigh with circulation

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## Abstract

In this paper, we consider the motion of the hydrodynamic Chaplygin sleigh, a planar rigid body in a potential flow with circulation around the body, subject to a nonholonomic constraint modeling a fin or keel attached to the body. We show that the motion of this system can be described by Euler-Poincaré-Suslov equations on a central extension of the special Euclidian group  $SE(2)$ , where the cocycle used to construct the extension encodes the effects of circulation upon the body. In the second part of the paper, we then discuss nonholonomic systems on central extensions of Lie groups, where both the Lagrangian and the nonholonomic constraints are left invariant. We show that there is a one-to-one correspondence between invariant measures on the original group and on the extended group, and we use this result to characterize the existence of an invariant measure for the hydrodynamic Chaplygin sleigh. We finish with a qualitative discussion of the reduced dynamics.

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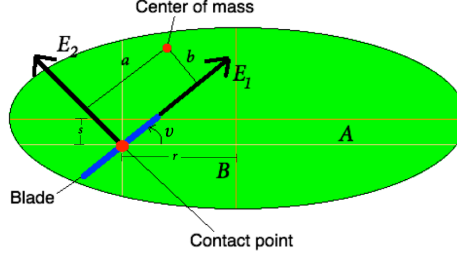
## 1 Introduction and outline

In this paper, we consider the motion of a planar rigid body surrounded by an irrotational perfect fluid with a given amount of circulation around the body. In addition, we assume that the body is equipped with a very effective keel or fin. To a good approximation, this fin imposes restrictions on the possible velocities of the body, and can therefore modeled by a nonholonomic constraint. In this way we obtain a low-dimensional model of an underwater vehicle with a rigid fin, which we have termed the *hydrodynamic Chaplygin sleigh with circulation*, after [10] where the corresponding model without circulation was described.

**History and overview of fluid-structure interactions.** The motion of a rigid body in a potential fluid in the absence of external forces was first described by Kirchhoff [17]. His crucial observation was that the effect of the fluid on the body could be described entirely in terms of the added mass and added inertia terms, which depend on the geometry of the body only, and can be calculated analytically for a wide class of body shapes. Kirchhoff’s solution was extended to the case of rigid bodies moving in potential flow with circulation by, among others, Chaplygin [6] and Lamb [19], who derived the equations of motion for this system, provided an explicit integration in terms of elliptic functions, and described qualitative features of the dynamics. In recent years, these ideas have been extended to the case of rigid bodies interacting with point vortices [31, 1], vortex rings [32] and other vortical structures, and they have been used to describe underwater vehicles [20] and the motion of bio-organisms [4, 5]. A comprehensive overview of the history of these equations can be found in [2]. Fluid-structure models involving nonholonomic constraints have been considered in the aerospace engineering community [28], while robotic models for underwater vehicles taking into account the effects of circulation were considered in [15].

**The hydrodynamic Chaplygin sleigh.** The model described in this paper incorporates nonholonomic constraints into the Chaplygin-Lamb system described above. We refer to this model as the hydrodynamic Chaplygin sleigh, since in the absence of the fluid the nonholonomic constraint models the effect of a sharp blade in the classical Chaplygin sleigh problem [7] which prevents the sleigh from moving in the lateral direction. It is an interesting historic coincidence that the name of Chaplygin is linked both to the development of the Chaplygin-Lamb equations [6] as well as to the nonholonomic Chaplygin sleigh [7].

Consider a body frame  $\{\mathbf{E}_1, \mathbf{E}_2\}$  where  $\mathbf{E}_1$  is aligned with the fin of the body (see Figure 1). The



**Figure 1:** The hydrodynamic Chaplygin sleigh consists of a rigid body moving in a potential fluid equipped with a very effective fin or blade, which prevents motion in the direction lateral to the fin. The body frame  $\{\mathbf{E}_1, \mathbf{E}_2\}$  is taken so that  $\mathbf{E}_1$  is aligned with the fin. The fin does not have to be aligned with the axes of the body, and we make no assumptions about the relative location of the center of mass, the geometric center of the body, and the contact point of the fin. We have depicted an elliptical body shape, but our results are valid for arbitrary convex shapes.

equations of motion for the hydrodynamic Chaplygin sleigh with circulation have the following form:

$$\begin{aligned}\dot{k} &= v_2 p_1 - v_1 p_2 - \rho(\alpha v_1 + \beta v_2), \\ \dot{p}_1 &= \omega p_2 - \kappa \rho v_2 + \rho \alpha \omega, \\ \dot{p}_2 &= -\omega p_1 + \kappa \rho v_1 + \rho \beta \omega + \lambda.\end{aligned}\tag{1.1}$$

Here,  $\rho$  is the density of the fluid,  $(\omega, v_1, v_2)$  are the angular and linear velocities of the body,  $(k, p_1, p_2)$  represent the corresponding impulses defined via the added inertia tensor,  $\kappa$  and  $\alpha, \beta$  are constants related to circulation, and  $\lambda$  is a Lagrange multiplier determined uniquely from the nonholonomic constraint

$$v_2 = 0,\tag{1.2}$$

which precludes motion in the direction perpendicular to the blade. We emphasize that these equations are valid for *arbitrary body shapes*; the geometry of the body is encoded in the added inertia matrix of the body (see (2.10) below).

If the constraint is not enforced and its Lagrange multiplier is set to zero, the corresponding equations become those of Chaplygin and Lamb [19]. If instead the circulation vanishes ( $\kappa = 0$ , and as a result,  $\alpha = \beta = 0$ ), the resulting system is the hydrodynamic Chaplygin sleigh treated in [10].

**Hamiltonian structure of the Chaplygin-Lamb equations.** The equations (1.1) have several interesting features. In section 3 we begin by focusing on the Chaplygin-Lamb equations: in other words, we consider a rigid planar body of arbitrary shape immersed in a potential flow with circulation, but we do not yet take the nonholonomic constraint (1.2) into account. In section 3.2 we show that the Chaplygin-Lamb equations are Hamiltonian with respect to the following Poisson bracket, defined on functions  $F, K$  of the angular and linear momenta  $\mu = (k, p_1, p_2)$ :

$$\{F, K\}_{\kappa, \alpha, \beta}(\mu) = (\nabla_\mu F)^T \begin{pmatrix} 0 & -p_2 & p_1 \\ p_2 & 0 & 0 \\ -p_1 & 0 & 0 \end{pmatrix} (\nabla_\mu K) - \begin{pmatrix} \rho \kappa \\ -\rho \beta \\ \rho \alpha \end{pmatrix} \cdot ((\nabla_\mu F) \times (\nabla_\mu K)).\tag{1.3}$$

The first term in this expression is the Lie-Poisson bracket on the dual space  $\mathfrak{se}(2)^*$ , while the second term is non-canonical, and is due to the presence of circulation.

In section 3.3, we make a digression to give a geometric interpretation of the constants  $\alpha, \beta$ , and in particular their dependence on the circulation  $\kappa$ . While these constants were introduced already by Lamb [19], their role remained somewhat mysterious. We introduce the *cyclic centroid* of a rigid body with circulation, and we show that  $\alpha$  and  $\beta$  can be made zero by attaching the body frame to the cyclic centroid, leading to a corresponding simplification in the equations (1.1). Geometrically, the cyclic centroid is the center of the circle which best approximates the circumference of the rigid body, in the sense of complex approximation theory [9] (see Figure 4). It can then be shown that the cyclic centroid is the root (in the complex plane) of the first Faber polynomial of the rigid body, while the radius of the best-fitting circle is given by the conformal capacity of the rigid body.

**Circulation in terms of central extensions.** In section 4 we show that the noncanonical Poisson bracket (1.3), defined on the dual of the Lie algebra of the Lie group  $\text{SE}(2)$ , can be made into a canonical Lie-Poisson bracket by considering a *central extension* of  $\text{SE}(2)$  by  $\mathbb{R}^3$ . The  $\mathbb{R}^3$ -valued cocycle used to construct this central extension is shown to consist of two parts: the first is an  $\mathbb{R}$ -valued cocycle which models the effect of non-zero circulation  $\kappa$ . This cocycle was introduced in [35] and gives rise to the *oscillator group*, a non-trivial central extension of  $\text{SE}(2)$  by  $\mathbb{R}$ . The second cocycle takes values in  $\mathbb{R}^2$  and models the effect of the coefficients  $\alpha, \beta$  in the Chaplygin-Lamb equations. In contrast to the first cocycle, this cocycle is exact, and we show that it can be “gauged away” by attaching the body frame of reference to the cyclic centroid. When nonholonomic constraints are present, this simplification is often undesirable, as it would lead to more complicated expressions for the constraints, so that we have to include the effect of nonzero  $\alpha, \beta$ .

**Nonholonomic mechanics on central extensions.** In section 5 we incorporate nonholonomic constraints into our model. We begin by describing the class of *LL systems on central extensions*, nonholonomic mechanical systems for which the configuration space is the central extension of a Lie group, and where both the Lagrangian and the constraints are invariant under the left action of the extended group on itself. In the case of the equations (1.1), the underlying Lie group is the central extension of  $\text{SE}(2)$  by  $\mathbb{R}^3$  described previously, and the LL invariance expresses the invariance of the equations (1.1) and the constraint (1.2) under translations and rotations of the rigid body and the surrounding fluid.

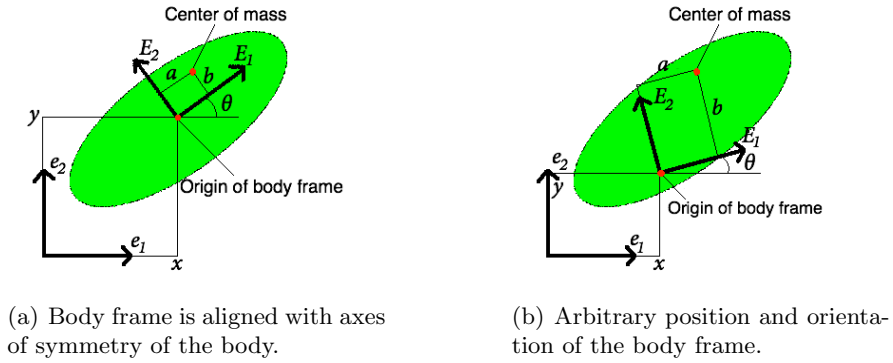
Given a Lagrangian and a set of nonholonomic constraints on a central extension  $\hat{G}$  of Lie group  $G$ , we derive the corresponding reduced equations of motion, referred to as the *Euler-Poincaré-Suslov equations*. We then turn to the question of whether these equations have an *invariant measure*. This problem has been considered for general EPS equations on Lie algebras of compact groups by Kozlov [18], by Jovanović [13] in the non-compact case, and by Zenkov and Bloch [36] for systems with nontrivial shape space. Using the criterion of Jovanović, we show that the EPS equations on a central extension  $\hat{G}$  admit an invariant measure if and only if the corresponding system on the original group  $G$  has an invariant measure, and we use this result to characterize the existence of invariant measures for the equations (1.1) in terms of the added moments of inertia of the hydrodynamic Chaplygin sleigh. In particular, we obtain that the existence of an invariant measure is independent of the value of the circulation  $\kappa$  and the constants  $\alpha, \beta$ .

**Qualitative features of the dynamics.** We finish the paper with an overview of some features of the reduced dynamics. In particular, we show that there exists a one-parameter family of equilibria in the reduced phase space, and that the other solutions are asymptotic motions between the equilibria. A more detailed description of the reduced dynamics, including analytic expressions for the solutions, will be given in [11].

## 2 Kinematics of rigid bodies and ideal fluids

We adopt Euler's approach to the study of the rigid body dynamics and consider an orthonormal *body frame*  $\{\mathbf{E}_1, \mathbf{E}_2\}$  that is attached to the body. This frame is related to a fixed *space frame*  $\{\mathbf{e}_1, \mathbf{e}_2\}$  by a rotation by an angle  $\theta$  that specifies the orientation of the two dimensional body at each time. We will denote by  $\mathbf{x} = (x, y) \in \mathbb{R}^2$  the spatial coordinates of the origin of the body frame (see Figure 2). The configuration of the body at any time is completely determined by the element  $g$  of the two dimensional Euclidean group  $\text{SE}(2)$  given by

$$g = \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SE}(2).$$



**Figure 2:** Two different choices of the body frame for an elliptical two-dimensional rigid body. In both cases the origin of the body frame does not coincide with the center of mass.

We will often denote the above element in  $g \in \text{SE}(2)$  by  $g = (R_\theta, \mathbf{x})$ , where  $R_\theta \in \text{SO}(2)$  is the rotation matrix determined by the angle  $\theta$ . Let  $(v_1, v_2) \in \mathbb{R}^2$  be the linear velocity of the origin of the body frame written in the body coordinates, and denote by  $\omega = \dot{\theta}$  the body's angular velocity. They define the element  $\xi$  in the Lie algebra  $\mathfrak{se}(2)$  given by

$$\xi = g^{-1}\dot{g} = \begin{pmatrix} 0 & -\omega & v_1 \\ \omega & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{se}(2). \quad (2.4)$$

Explicitly we have

$$\dot{\theta} = \omega, \quad v_1 = \dot{x} \cos \theta + \dot{y} \sin \theta, \quad v_2 = -\dot{x} \sin \theta + \dot{y} \cos \theta. \quad (2.5)$$

For convenience, we will sometimes identify  $\mathfrak{se}(2)$  with  $\mathbb{R}^3$  as vector spaces, and denote  $\xi \in \mathfrak{se}(2)$  as the column vector  $(\omega, v_1, v_2)^T \in \mathbb{R}^3$ . The Lie algebra commutator takes the form

$$[(\omega, v_1, v_2), (\omega', v'_1, v'_2)]_{\mathfrak{se}(2)} = (0, v_2\omega' - \omega v'_2, \omega v'_1 - v_1\omega').$$

The kinetic energy of the body is given by

$$T_{\mathcal{B}} = \frac{1}{2} ((\mathcal{I}_{\mathcal{B}} + ma^2 + mb^2)\omega^2 + m(v_1^2 + v_2^2) - mb\omega v_1 + mawv_2),$$

where  $m$  is the mass of the body,  $(a, b)$  are body coordinates of the center of mass (see Figure 2), and  $\mathcal{I}_{\mathcal{B}}$  is the moment of inertia of the body about the center of mass. It is a positive definite quadratic form on  $\mathfrak{se}(2)$  whose matrix is the *body inertia tensor*

$$\mathbb{I}_{\mathcal{B}} = \begin{pmatrix} \mathcal{I}_{\mathcal{B}} + m(a^2 + b^2) & -mb & ma \\ -mb & m & 0 \\ ma & 0 & m \end{pmatrix}. \quad (2.6)$$

For future reference, we give an explicit description of the dual space  $\mathfrak{se}(2)^*$ . Since  $\mathfrak{se}(2)$  is isomorphic to  $\mathbb{R}^3$  and using the Euclidian inner product, we have that  $\mathfrak{se}(2)^* \cong \mathbb{R}^3$ . A typical element  $\mu$  is represented as a row vector  $\mu = (k, p_1, p_2)$ . The duality pairing between  $\mu$  and an element  $\xi = (\omega, v_1, v_2)^T$  of  $\mathfrak{se}(2)$  is given by

$$\langle \mu, \xi \rangle = \mu \xi = k\omega + p_1 v_1 + p_2 v_2.$$

Lastly, we also mention that  $\text{SE}(2)$  acts on  $\mathfrak{se}(2)$  and  $\mathfrak{se}(2)^*$  by the *adjoint* and *coadjoint actions*, respectively. For the present paper, we will often need the infinitesimal version of the coadjoint action. This action is denoted by  $\text{ad}_{\xi}^* : \mathfrak{se}(2)^* \rightarrow \mathfrak{se}(2)^*$  for all  $\xi \in \mathfrak{se}(2)$  and is given explicitly by

$$\text{ad}_{\xi}^*(\mu) = (p_1 v_2 - p_2 v_1, \omega p_2, -\omega p_1), \quad (2.7)$$

where  $\xi = (\omega, v_1, v_2)^T$  and  $\mu = (k, p_1, p_2)$ .

The dual space  $\mathfrak{se}(2)^*$  is equipped with the (minus) *Lie-Poisson bracket*, which is given by

$$\{F, K\}_{\mathfrak{se}(2)^*}^-(\mu) = - \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle$$

for all functions  $F, K$  on  $\mathfrak{se}(2)^*$ . In coordinates, we have

$$\{F, K\}_{\mathfrak{se}(2)^*}^-(\mu) = (\nabla_{\mu} F)^T \begin{pmatrix} 0 & -p_2 & p_1 \\ p_2 & 0 & 0 \\ -p_1 & 0 & 0 \end{pmatrix} (\nabla_{\mu} K),$$

where  $\nabla_{\mu} F$  is the gradient of  $F$  with respect to the variables  $(k, p_1, p_2) = \mu$ .

**The fluid flow at a given instant.** Consider now the motion of the fluid that surrounds the body. Suppose that at a given instant the body occupies a region  $\mathcal{B} \subset \mathbb{R}^2$ . The flow is assumed to take place in the connected unbounded region  $\mathcal{U} := \mathbb{R}^2 \setminus \mathcal{B}$  that is not occupied by the body. We assume that the flow is potential so the Eulerian velocity of the fluid  $\mathbf{u}$  can be written as  $\mathbf{u} = \nabla \Phi$  for a fluid potential  $\Phi : \mathcal{U} \rightarrow \mathbb{R}$ . Incompressibility of the fluid implies that  $\Phi$  is harmonic,

$$\nabla^2 \Phi = 0 \quad \text{on} \quad \mathcal{U}.$$

The boundary conditions for  $\Phi$  come from the following considerations. On the one hand it is assumed that, up to a purely circulatory flow around the body, the motion of the fluid is solely due to the motion of the body. This assumption requires the fluid velocity  $\nabla \Phi$  to vanish at infinity. Secondly, to avoid cavitation or penetration of the fluid into the body, we require the normal component of the fluid velocity at a material point  $p$  on the boundary of  $\mathcal{B}$  to agree with the normal component of the velocity of  $p$ . Suppose that the vector  $(X, Y) \in \mathbb{R}^2$  gives body coordinates for  $p$ . The latter boundary condition is expressed as

$$\left. \frac{\partial \Phi}{\partial n} \right|_{p \in \partial \mathcal{B}} = (v_1 - \omega Y)n_1 + (v_2 + \omega X)n_2,$$

where  $\mathbf{n} = (n_1, n_2)$  is the outward unit normal vector to  $\mathcal{B}$  at  $p$  written in body coordinates. These conditions determine the flow of the fluid up to a purely circulatory flow around the body that would persist if the body is brought to rest. The latter is specified by the value of the circulation  $\kappa$  around the body as we now discuss.

The potential  $\Phi$  that satisfies the above boundary value problem can be written in terms of the body's velocities  $v_1, v_2, \omega$ , in *Kirchhoff form*:

$$\Phi = v_1 \phi_1 + v_2 \phi_2 + \omega \chi + \phi_0, \tag{2.8}$$

where  $\phi_i$ ,  $i = 0, 1, 2$ , and  $\chi$  are harmonic functions on  $\mathcal{U}$  whose gradients vanish at infinity and satisfy:

$$\left. \frac{\partial \phi_i}{\partial n} \right|_{\partial \mathcal{B}} = n_i, \quad i = 1, 2, \quad \left. \frac{\partial \chi}{\partial n} \right|_{\partial \mathcal{B}} = Xn_2 - Yn_1, \quad \left. \frac{\partial \phi_0}{\partial n} \right|_{\partial \mathcal{B}} = 0.$$

The potential  $\phi_0$  is multi-valued and defines the circulatory flow around the body. The circulation  $\kappa$  of the fluid around the body satisfies

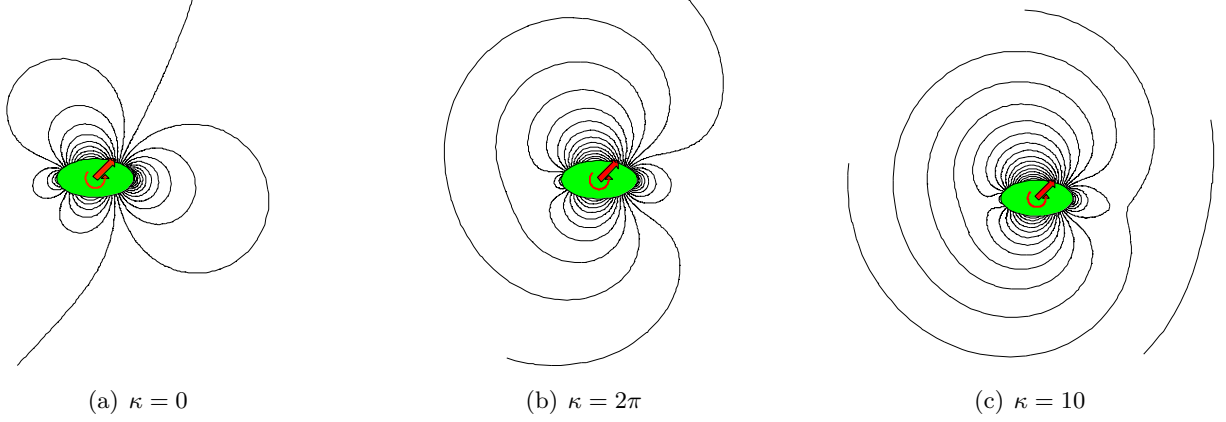
$$\kappa = \oint_{\partial \mathcal{B}} \mathbf{u} \cdot d\mathbf{l} = \oint_{\partial \mathcal{B}} \nabla \phi_0 \cdot d\mathbf{l}, \tag{2.9}$$

and remains constant during the motion. Figure 3 shows the streamline pattern of the flow determined by the motion of an elliptical body for different values of  $\kappa$ .

Disregarding the circulatory motion, the kinetic energy of the fluid is given by

$$T_{\mathcal{F}} = \frac{\rho}{2} \int_{\mathcal{U}} \|\nabla(\Phi - \phi_0)\|^2 dA,$$

where  $dA$  is the area element in  $\mathbb{R}^2$  and  $\rho$  is the (constant) fluid density. We have subtracted the circulatory part from the velocity potential, as it would give rise to an infinite contribution to the fluid kinetic energy [30].



**Figure 3:** Stream line pattern for an ellipse moving on the plane for different values of the circulation  $\kappa$ . The major and minor semi-axes of the ellipse are  $A = 2, B = 1$ . The body frame is aligned with the principal axes of the ellipse and the velocity of the body satisfies  $v_1 = v_2 = 1$  and  $\omega = 5$ .

By substituting (2.8) into the above, one can express  $T_{\mathcal{F}}$  as the quadratic form

$$T_{\mathcal{F}} = \frac{1}{2} \left( \sum_{i,j=1}^2 \mathcal{M}_{\mathcal{F}}^{ij} v_i v_j + 2 \sum_{i=1}^2 \mathcal{K}_{\mathcal{F}}^i v_i \omega + \mathcal{I}_{\mathcal{F}} \omega^2 \right), \quad (2.10)$$

where  $\mathcal{M}_{\mathcal{F}}^{ij}, \mathcal{K}_{\mathcal{F}}^i, i, j = 1, 2$ , and  $\mathcal{I}_{\mathcal{F}}$  are certain constants that only depend on the body shape. Explicitly one has (see [19] for details),

$$\begin{aligned} \mathcal{M}_{\mathcal{F}}^{ij} &= -\rho \int_{\partial \mathcal{B}} \phi_i \frac{\partial \phi_j}{\partial n} dl = -\rho \int_{\partial \mathcal{B}} \phi_j \frac{\partial \phi_i}{\partial n} dl, \quad i, j = 1, 2, & \mathcal{I}_{\mathcal{F}} &= -\rho \int_{\partial \mathcal{B}} \chi \frac{\partial \chi}{\partial n} dl \\ \mathcal{K}_{\mathcal{F}}^i &= -\rho \int_{\partial \mathcal{B}} \phi_i \frac{\partial \chi}{\partial n} dl = -\rho \int_{\partial \mathcal{B}} \chi \frac{\partial \phi_i}{\partial n} dl, \quad i = 1, 2. \end{aligned}$$

These constants are referred to as *added masses* and are conveniently written in  $3 \times 3$  matrix form to define the (symmetric) *added inertia tensor*:

$$\mathbb{I}_{\mathcal{F}} := \begin{pmatrix} \mathcal{I}_{\mathcal{F}} & \mathcal{K}_{\mathcal{F}} \\ \mathcal{K}_{\mathcal{F}}^T & \mathcal{M}_{\mathcal{F}} \end{pmatrix},$$

that defines  $T_{\mathcal{F}}$  as a quadratic form on  $\mathfrak{se}(2)$ .

**Example.** For an elliptic rigid body with semi-axes of length  $A > B > 0$ , the added masses and moments of inertia take on a particularly convenient form. The kinetic energy of the fluid is given by (see [19])

$$T_{\mathcal{F}} = \frac{\rho\pi}{2} \left( B^2 v_1^2 + A^2 v_2^2 + \frac{(A^2 - B^2)^2}{4} \omega^2 \right),$$



where we have ignored the circulatory motion around the body. The corresponding added inertia tensor is thus given by

$$\mathbb{I}_{\mathcal{F}} = \rho\pi \begin{pmatrix} \frac{(A^2-B^2)^2}{4} & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & A^2 \end{pmatrix}. \quad (2.11)$$

We emphasize that this particular form of the added inertia tensor was derived under the assumption that the body frame is aligned with the symmetry axes of the ellipse ( $\nu = 0$ ,  $r = s = 0$  in Figure 1). When this is not the case, the added mass tensor is more complicated, and in particular need not be diagonal, as is shown in (5.40).

### 3 Hamiltonian dynamics of rigid bodies in potential flow

In this section, we describe the Kirchhoff equations for the dynamics of a rigid body in potential flow, and we review the Chaplygin-Lamb equations dealing with rigid bodies in the presence of circulation. We show how, in the absence of circulation, the equations of motion can be viewed as Lie-Poisson equations on the space  $\mathfrak{se}(2)^*$ , the dual of the Lie algebra of the Lie group  $SE(2)$  of translations and rotations in the plane. In the presence of circulation, new gyroscopic forces enter the equations and the Hamiltonian description of the equations involves a non-canonical Poisson bracket on  $\mathfrak{se}(2)^*$ .

In section 3.3, we define the *cyclic centroid* of a rigid body with circulation and we show how this distinguished point can be described in terms of conformal geometry of the body. Apart from the material in the last section and the discussion of the Poisson structures, which is new (to the best of our knowledge), most of the material covered in the first two sections can be found, for instance, in [26, 14] as well as in the classical works of Lamb [19] and Milne-Thomson [24].

#### 3.1 Rigid body motion in potential flow

The total kinetic energy,  $T$ , of the solid-fluid system (excluding the circulatory motion) is the sum of the kinetic energy  $T_{\mathcal{B}}$  of the rigid body and the energy  $T_{\mathcal{F}}$  of the fluid. As both  $T_{\mathcal{B}}$  and  $T_{\mathcal{F}}$  are quadratic functions on  $\mathfrak{se}(2)$ , so is the total energy  $T$ . In the absence of external forces or circulation, the Lagrangian  $L$  of the solid-fluid system is just the kinetic energy:  $L = T$ , and the associated equations of motion are the so-called *Euler-Poincaré equations* [23] on  $\mathfrak{se}(2)$ . These equations determine the evolution of the body linear and angular velocity  $\xi = (\omega, v_1, v_2)$ , and from the *reconstruction equations* (2.5), the position and orientation of the body can then be obtained as a function of time. The latter determine a curve in  $SE(2)$ , and it can be shown that this curve is a geodesic with respect to the (left-invariant) kinetic energy metric.

The function  $L$  on  $\mathfrak{se}(2)$  is called the *reduced Lagrangian*. Explicitly, it is given by

$$L(\xi) = \frac{1}{2} \xi^T \mathbb{I} \xi,$$

where  $\xi = (\omega, v_1, v_2)^T \in \mathfrak{se}(2) \cong \mathbb{R}^3$  is thought as a column vector and the matrix  $\mathbb{I}$  is the sum of the inertia matrix  $\mathbb{I}_{\mathcal{B}}$  of the rigid body and the added masses and inertia  $\mathbb{I}_{\mathcal{F}}$  of the fluid:  $\mathbb{I} = \mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}}$ .

For future reference, we note that  $L$  induces a function  $\mathcal{L}$  on  $T\text{SE}(2)$  by left translation, defined by  $\mathcal{L}(g, \dot{g}) = L(g^{-1}\dot{g})$ ; see also (2.4).

We now focus on the Hamiltonian formulation of the equations of motion for a rigid body in potential flow. Recall that a typical element of the dual  $\mathfrak{se}(2)^*$  is written as a row vector  $\mu = (k, p_1, p_2)$  and represents the angular and linear momenta of the rigid body (in the body frame). With this identification, the *Legendre transform* associated to  $L$  is defined as the mapping

$$\mathbb{F}L : \mathfrak{se}(2) \rightarrow \mathfrak{se}(2)^*$$

given by  $\mathbb{F}L(\xi) = \mu$ , where  $\mu = (\mathbb{I}\xi)^T$ . With the notations of Figure 2, the components of  $\mu = (k, p_1, p_2)$  are explicitly given by

$$\begin{aligned} k &= (\mathcal{I}_{\mathcal{B}} + m(a^2 + b^2) + \mathcal{I}_{\mathcal{F}})\omega + (-mb + \mathcal{K}_{\mathcal{F}}^1)v_1 + (ma + \mathcal{K}_{\mathcal{F}}^2)v_2, \\ p_1 &= (-mb + \mathcal{K}_{\mathcal{F}}^1)\omega + (m + \mathcal{M}_{\mathcal{F}}^{11})v_1 + \mathcal{M}_{\mathcal{F}}^{12}v_2, \\ p_2 &= (ma + \mathcal{K}_{\mathcal{F}}^2)\omega + \mathcal{M}_{\mathcal{F}}^{12}v_1 + (m + \mathcal{M}_{\mathcal{F}}^{22})v_2. \end{aligned} \tag{3.12}$$

In classical hydrodynamics  $k$  and  $(p_1, p_2)$  are known as “impulsive pair” and “impulsive force” respectively. The reduced Hamiltonian  $H : \mathfrak{se}(2)^* \rightarrow \mathbb{R}$  is given by

$$H(\mu) = \frac{1}{2}\mu\mathbb{I}^{-1}\mu^T, \tag{3.13}$$

and the corresponding (minus) Lie-Poisson equations are  $\dot{\mu} = \text{ad}_{\mathbb{I}^{-1}\mu}^*\mu$ , where the infinitesimal coadjoint action is given by (2.7). Written out in component form, these equations are nothing but the *Kirchhoff equations*:

$$\begin{aligned} \dot{k} &= v_2p_1 - v_1p_2, \\ \dot{p}_1 &= \omega p_2, \quad \dot{p}_2 = -\omega p_1, \end{aligned} \tag{3.14}$$

where the velocities  $(\omega, v_1, v_2)^T$  and the impulses  $(k, p_1, p_2)$  are related by the Legendre transformation (3.12). Finally, we remark that the motion of the body in space can be found from a solution of (3.14) by solving the *reconstruction equations* (2.5).

### 3.2 Rigid body motion with circulation

In the presence of circulation, the Kirchhoff equations on  $\mathfrak{se}(2)^*$  have to be modified to include the Kutta-Zhukowski force. This is a gyroscopic force, which is proportional to the circulation  $\kappa$ . In this case, the equations of motion are referred to as the *Chaplygin-Lamb equations* [6, 19], and they are given by

$$\begin{aligned} \dot{k} &= v_2p_1 - v_1p_2 - \rho(\alpha v_1 + \beta v_2), \\ \dot{p}_1 &= \omega p_2 - \kappa\rho v_2 + \rho\alpha\omega, \\ \dot{p}_2 &= -\omega p_1 + \kappa\rho v_1 + \rho\beta\omega, \end{aligned} \tag{3.15}$$

The constants  $\alpha$  and  $\beta$  are proportional to the circulation  $\kappa$  and depend on the position and orientation of the body axes. They are explicitly given by:

$$\alpha = \oint_{\partial\mathcal{B}} X \nabla \phi_0 \cdot d\mathbf{l}, \quad \beta = \oint_{\partial\mathcal{B}} Y \nabla \phi_0 \cdot d\mathbf{l}, \quad (3.16)$$

where, as before,  $(X, Y)$  are body coordinates for material points in  $\partial\mathcal{B}$ . The Chaplygin-Lamb equations are discussed in detail in [2] and a derivation from first principles, using techniques from symplectic geometry and reduction theory, is given in [35].

One easily verifies that if the center of the body axes is displaced to the point with body coordinates  $(r, s)$ , so that the new body coordinates are  $\tilde{X} = X - r$ ,  $\tilde{Y} = Y - s$ , then the circulation constants relative to the new coordinate axes take the form  $\tilde{\alpha} = \alpha - r\kappa$ ,  $\tilde{\beta} = \beta - s\kappa$ . Thus, there is a unique choice of the body axes that makes these constants vanish. On the other hand, it is also often desirable to choose the body axes so that the total inertia tensor  $\mathbb{I}$  is diagonal. For an asymmetric body, these two choices are in general incompatible, see e.g. [19].

For our purposes, the choice of body axes will be made to simplify the expression of the nonholonomic constraint. We therefore consider equations (3.15) in their full generality where  $\alpha, \beta \neq 0$ , and  $\mathbb{I}$  is not diagonal. This contrasts with the treatment in [35] where it is assumed that  $\alpha = \beta = 0$  and with [2] where the complementary assumption, namely that  $\mathbb{I}$  is diagonal, is made.

A calculation shows that equations (3.15) are Hamiltonian with respect to the usual Hamiltonian  $H : \mathfrak{se}(2)^* \rightarrow \mathbb{R}$  given in (3.13) and with respect to the following bracket of functions on  $\mathfrak{se}(2)^*$ :

$$\{F, K\}_{\kappa, \alpha, \beta}(\mu) = (\nabla_\mu F)^T \begin{pmatrix} 0 & -p_2 & p_1 \\ p_2 & 0 & 0 \\ -p_1 & 0 & 0 \end{pmatrix} (\nabla_\mu K) - \begin{pmatrix} \rho\kappa \\ -\rho\beta \\ \rho\alpha \end{pmatrix} \cdot ((\nabla_\mu F) \times (\nabla_\mu K)). \quad (3.17)$$

where, “ $\times$ ” denotes the standard vector product in  $\mathbb{R}^3$ , and, as before,  $\nabla_\mu F$  is the gradient of  $F$  with respect to the variables  $(k, p_1, p_2) = \mu$ . In principle one would need to show that the above bracket satisfies the Jacobi identity. This will follow from the results of section 4 where it will be shown that the above bracket is Lie-Poisson on the dual Lie algebra of a central extension of  $\text{SE}(2)$  by  $\mathbb{R}^3$ .

The first term in (3.17) is the usual (minus) Lie-Poisson bracket on  $\mathfrak{se}(2)^*$  and defines the dynamics in the absence of circulation. The second term, which vanishes in the absence of circulation, is the remnant of the magnetic term in the reduced form in  $T^*\text{SE}(2)$ . For future reference, we record here the form of the bracket for  $\alpha = \beta = 0$ :

$$\{F, K\}_\kappa(\mu) = (\nabla_\mu F)^T \begin{pmatrix} 0 & -p_2 & p_1 \\ p_2 & 0 & 0 \\ -p_1 & 0 & 0 \end{pmatrix} (\nabla_\mu K) - \rho\kappa \left( \frac{\partial F}{\partial p_1} \frac{\partial K}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial K}{\partial p_1} \right). \quad (3.18)$$

### 3.3 Conformal geometry and the constants $\alpha, \beta$

In this section, we describe some of the complex geometry underlying the constants  $\alpha$  and  $\beta$  defined in (3.16). As discussed in the previous section, there exists a unique choice of origin for the body frame relative to which  $\alpha$  and  $\beta$  vanish. We denote this point by  $\mathbf{r}_c = (x_c, y_c)$  and refer to it as the *cyclic*

centroid. Using (2.9) and (3.16) one deduces

$$\mathbf{r}_c = \frac{1}{\kappa} \oint_{\partial\mathcal{B}} \mathbf{r} \frac{\partial\phi_0}{\partial l} dl.$$

In general, the location of this point will be different from the geometric centroid of the body, but it shares one important characteristic with the centroid: it can be found among the fixed points of the isometries of the body, if any. As an easy consequence, we have that for a homogeneous rigid body with two symmetry axes, the cyclic centroid, the geometric centroid, and the center of mass all coincide. For asymmetric bodies, the three points will in general be different.

**Proposition 3.1.** *Let  $T : \mathcal{B} \rightarrow \mathcal{B}$  be an isometry of the body, i.e. a linear isomorphism of  $\mathbb{R}^2$  such that  $T(\mathcal{B}) = \mathcal{B}$ . Then  $\mathbf{r}_c$  is a fixed point of  $T$ .*

*Proof.* Using the linearity of  $T$  we have

$$T(\mathbf{r}_c) = \frac{1}{\kappa} \oint_{\partial\mathcal{B}} T(\mathbf{r}) \frac{\partial\phi_0}{\partial l} dl = \frac{1}{\kappa} \oint_{T^{-1}(\partial\mathcal{B})} \mathbf{r}' \frac{\partial\phi_0}{\partial l'} dl' = \frac{1}{\kappa} \oint_{\partial\mathcal{B}} \mathbf{r}' \frac{\partial\phi_0}{\partial l'} dl',$$

since  $T$  leaves  $\partial\mathcal{B}$  invariant, and therefore  $T(\mathbf{r}_c) = \mathbf{r}_c$ .  $\square$

The location of the cyclic centroid can be made more explicit using some methods from complex geometry. Let  $W(z) = \phi_0 + i\psi_0$  be the complex potential associated to the cyclic motion and put  $z_c = x_c + iy_c$ . We then have that along the boundary

$$\frac{\partial\phi_0}{\partial l} dl = \Re \left( \frac{dW}{dz} dz \right) \quad \text{and} \quad \frac{\partial\psi_0}{\partial l} dl = \Im \left( \frac{dW}{dz} dz \right)$$

but since the boundary of the body is a streamline, the imaginary part of  $W(z)$  is constant on the boundary, and hence

$$z_c = \frac{1}{\kappa} \oint_{\partial\mathcal{B}} z \Re \left( \frac{dW}{dz} dz \right) = \frac{1}{\kappa} \oint z dW,$$

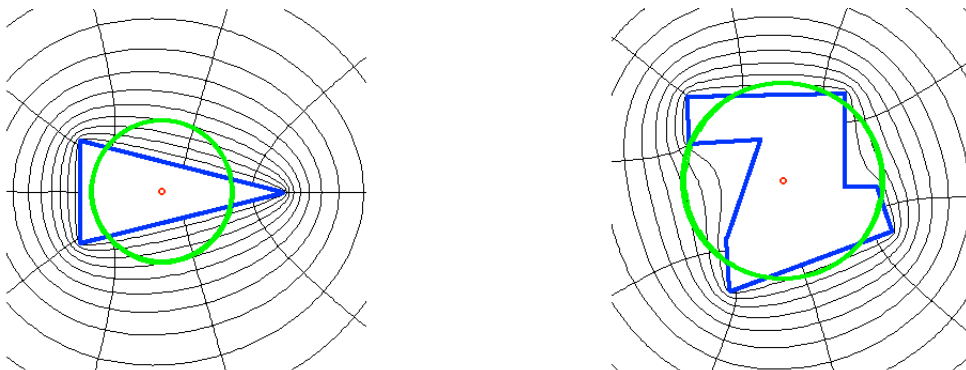
where the integral on the right-hand side is taken along an arbitrary contour around the body.

Now choose an arbitrary conformal map  $\zeta \mapsto z = z(\zeta)$  from the interior of the unit disc to the exterior of the rigid body, and assume that  $z(0) = \infty$ . The complex potential  $W(z)$  is related to the complex potential  $w(\zeta)$  in the  $\zeta$ -domain by  $W(z) = w(\zeta)$ , and moreover  $w(\zeta) = \frac{\kappa}{2\pi i} \log \zeta$ . As a result, the cyclic centroid can be expressed as a contour integral along the unit circle as

$$z_c = \frac{1}{\kappa} \oint_{|\zeta|=1} z(\zeta) dw = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{z(\zeta) d\zeta}{\zeta}.$$

Evaluating this contour integral requires us to know the constant term  $a_0$  in the Laurent series for  $z(\zeta)$  around 0. Since  $z(\zeta)$  only has a simple pole at infinity, we have that  $z_c = -a_0$ .

These manipulations hint at a neat geometric interpretation for  $z_c$  in terms of the first *Faber polynomial* and the *capacity* of the rigid body, viewed as a domain in the complex plane [8, 9]. Recall that the Faber polynomials of a complex domain  $D$  arise when considering a uniformizing map  $\Phi : z \mapsto$



**Figure 4:** Location of the cyclic center (red dot) for a triangular and an irregular body. This point is the center of the level set  $|f_1(z)| = 1$  (circle indicated in green) of the first Faber polynomial  $f_1(z)$ . The radius of this circle is the conformal capacity of the body.

$w := \Phi(z)$  from the exterior of  $D$  to the exterior of the unit disc. After suitable normalizations, this map has a Laurent expansion

$$\Phi(z) = \frac{1}{|C|}(z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots)$$

where  $C$  is the *capacity* of  $D$ . The  $m$ th Faber polynomial  $f_m(z)$  is then the polynomial part of  $[\Phi(z)]^m$ . In particular, we have that  $f_1(z) = |C|^{-1}(z + a_0)$ .

Now, note that the level set  $|\Phi(z)| = 1$  is precisely the boundary of the rigid body. In the theory of complex approximation, this level set is approximated by the level sets  $|f_m(z)| = 1$ , which provide successively better approximations as  $m$  increases. The first nontrivial level set,  $|f_1(z)| = 1$ , is a circle whose center is precisely the cyclic centroid  $z_c$  and its radius is the capacity  $|C|$  of the domain. From this point of view, the cyclic centroid is the center of the best-fitting circle through the boundary of the rigid body. In Figure 4, a few examples are shown of polygonal rigid bodies, for which  $z(\zeta)$  is a Schwartz-Christoffel mapping (see [9]). Also indicated is the level set of  $|f_1(z)| = 1$ .

## 4 Geodesic flow on central extensions of Lie groups

In this section, we give a geometric interpretation of the Chaplygin-Lamb equations. In particular, we show that the non-canonical bracket (3.17) on  $\mathfrak{se}(2)^*$  can be made into a canonical bracket on a “larger” space, obtained by considering a *central extension* of  $\text{SE}(2)$ . After reviewing the geometry of central extensions, we relate the particular extension, and the cocycles used in its description, to the circulation constants  $\kappa$ ,  $\alpha$ , and  $\beta$ .

After constructing the relevant central extension  $\widehat{G}$  of  $\text{SE}(2)$ , we show that the bracket (3.17) is precisely the Lie-Poisson bracket on the dual  $\widehat{\mathfrak{g}}^*$  of the Lie algebra, and we argue that the Chaplygin-Lamb equations hence give rise to geodesic flow on  $\widehat{G}$ . This result is interesting in its own right, and

provides the first step towards showing that the hydrodynamic Chaplygin sleigh with circulation can be formulated as an *LL-system*, i.e. a nonholonomic system on  $\widehat{G}$  whose Lagrangian and constraints are invariant under the left action of  $\widehat{G}$  on itself.

To construct the central extension  $\widehat{G}$ , we proceed in two steps. We first assume that  $\alpha = \beta = 0$  (recall that this can be achieved by a suitable choice of origin for the body-fixed frame), and we recall from [35] that the circulation  $\kappa$  is described by means of a real-valued cocycle  $C_1$ , giving rise to a central extension of  $\text{SE}(2)$  by  $\mathbb{R}$ , known as the *oscillator group* [33]. The effect of nonzero  $\alpha$  and  $\beta$  can then be described by a second cocycle  $C_2$  with values in  $\mathbb{R}^2$ , giving rise to a central extension of the oscillator group by  $\mathbb{R}^2$ . By combining both of these cocycles, we then describe  $\widehat{G}$  as a central extension of  $\text{SE}(2)$  by  $\mathbb{R}^3$ .

In terms of Lie algebra cohomology, the cocycle  $C_1$  is closed but not exact, while  $C_2$  will be shown to be exact. It is known that central extensions by exact cocycles are trivial, in the sense that the effect of  $C_2$  can be “gauged away” by a suitable redefinition of the momenta of the system (this is equivalent to the fact that  $\alpha$  and  $\beta$  can be made zero by a suitable choice of origin), while the circulation  $\kappa$  (described in terms of the non-exact cocycle  $C_1$ ) cannot be made to vanish.

#### 4.1 Central extensions of Lie groups

We briefly recall the definitions and basic properties of central extensions of Lie groups to introduce the relevant notation. A detailed account of the geometry of central extensions in the context of mechanics can be found in [22, 21, 16].

**Definition.** Let  $G$  be a Lie group and  $A$  an abelian Lie group. We will use additive notation for the group operation in  $A$ . Let  $\widehat{G}$  be an extension of  $G$  by  $A$ , that is, there is an exact sequence

$$0 \rightarrow A \rightarrow \widehat{G} \rightarrow G \rightarrow \{e\}.$$

Throughout the rest of this paper, we will assume that  $\widehat{G} = G \times A$  as a manifold and that the group multiplication on  $\widehat{G}$  is

$$(g, \alpha)(h, \beta) = (gh, \alpha + \beta + B(g, h)), \quad (4.19)$$

where  $B : G \times G \rightarrow A$  is a normalized group two-cocycle. Associativity of the multiplication on  $\widehat{G}$  is equivalent to the two-cocycle identity,

$$B(f, g) + B(fg, h) = B(f, gh) + B(g, h) \quad \text{for all } f, g, h \in G.$$

The assumption that the two-cocycle  $B$  is normalized can be made without loss of generality and amounts to

$$B(g, e) = B(e, g) = 0 \quad \text{for all } g \in G,$$

implying that  $B(g, g^{-1}) = B(g^{-1}, g)$ . Under these assumptions,  $\widehat{G}$  is referred to as a *central extension of  $G$  by  $A$* .

**Central extension of Lie algebras.** The Lie algebra  $\widehat{\mathfrak{g}}$  of  $\widehat{G}$  is isomorphic as a vector space to  $\mathfrak{g} \times \mathfrak{a}$ , and is equipped with the following bracket:

$$[(\xi, a), (\eta, b)]_{\widehat{\mathfrak{g}}} = ([\xi, \eta]_{\mathfrak{g}}, C(\xi, \eta))$$

for  $(\xi, a), (\eta, b) \in \widehat{\mathfrak{g}}$ . Here, the  $\mathfrak{a}$ -valued Lie algebra two-cocycle  $C : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{a}$  is defined by

$$C(\xi, \eta) := \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} (B(g(t), h(s)) - B(h(s), g(t))), \quad i = 1, 2, \quad (4.20)$$

with  $g(t), h(s)$  smooth curves on  $G$  satisfying  $\dot{g}(0) = \xi, \dot{h}(0) = \eta$ .

It may happen that  $C$  can be written in terms of a one-cocycle  $\mathcal{A} : \mathfrak{g} \rightarrow \mathfrak{a}$  as  $C(\xi, \eta) = -\mathcal{A}([\xi, \eta])$  for all  $\xi, \eta \in \mathfrak{g}$ . In this case,  $C$  is a *coboundary* in the sense of Lie algebra cohomology, and we write  $C = \delta\mathcal{A}$ . When this happens, the central extension is said to be *trivial*, as the mapping

$$\Psi_{\mathcal{A}} : \mathfrak{g} \times \mathfrak{a} \rightarrow \widehat{\mathfrak{g}}, \quad (\xi, a) \mapsto (\xi, a - \mathcal{A}(\xi)) \quad (4.21)$$

then determines a Lie algebra isomorphism between the Lie algebra  $\mathfrak{g} \times \mathfrak{a}$  with the product bracket and the central extension  $\widehat{\mathfrak{g}}$ .

**Lie-Poisson structures.** As a vector space, the dual Lie algebra  $\widehat{\mathfrak{g}}^*$  equals  $\mathfrak{g}^* \times \mathfrak{a}^*$ . For  $(\mu, \sigma) \in \widehat{\mathfrak{g}}^*$ , the  $\pm$  Lie-Poisson bracket of functions  $F, K \in C^\infty(\widehat{\mathfrak{g}}^*)$  is readily computed to be

$$\{F, K\}_{\widehat{\mathfrak{g}}^*}^\pm(\mu, \sigma) = \pm \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle \pm \left\langle \sigma, C \left( \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right) \right\rangle. \quad (4.22)$$

Notice that this bracket only involves functional derivatives with respect to  $\mu$ . Therefore, if we fix the value of  $\sigma \in \mathfrak{a}^*$  in (4.22), we obtain a non-canonical Poisson bracket on  $\mathfrak{g}^*$ , given by formally the same expression as (4.22): for  $f, k \in C^\infty(\mathfrak{g}^*)$  we have

$$\{f, k\}_\sigma^\pm(\mu) = \pm \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu} \right] \right\rangle \pm \left\langle \sigma, C \left( \frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu} \right) \right\rangle, \quad (4.23)$$

where  $\sigma \in \mathfrak{a}^*$  is now regarded as fixed. Roughly speaking, we therefore have a one-to-one correspondence between Poisson brackets on  $\mathfrak{g}^*$  which are the sum of a Lie-Poisson term and a cocycle, and Lie-Poisson brackets on central extensions  $\widehat{\mathfrak{g}}^*$ . This can be made rigorous by observing that the injection  $\iota_\sigma : \mathfrak{g}^* \hookrightarrow \widehat{\mathfrak{g}}^*$ , given by  $\iota_\sigma(\mu) := (\mu, \sigma)$  for  $\sigma$  fixed, is a Poisson map taking the non-canonical Poisson structure (4.23) into the Lie-Poisson structure (4.22).

In the case of a trivial central extension, the cocycle term in (4.23) can be “gauged away”: let  $C = \delta\mathcal{A}$ . For  $\sigma \in \mathfrak{a}^*$ , we denote by  $\mathcal{A}_\sigma : \mathfrak{g} \rightarrow \mathbb{R}$  the linear map defined by  $\mathcal{A}_\sigma(\xi) := \langle \sigma, \mathcal{A}(\xi) \rangle$  for all  $\xi \in \mathfrak{g}$ . Note that  $\mathcal{A}_\sigma \in \mathfrak{g}^*$ . The Poisson bracket (4.22) can then be rewritten as

$$\begin{aligned} \{F, K\}_{\widehat{\mathfrak{g}}^*}^\pm(\mu, \sigma) &= \pm \left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle \mp \left\langle \sigma, \mathcal{A} \left( \left[ \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right) \right\rangle \\ &= \pm \left\langle \mu - \mathcal{A}_\sigma, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle, \end{aligned}$$

so that the cocycle vanishes apart from a shift  $\mu \mapsto \mu - \mathcal{A}_\sigma$  in the momenta of the system.

**Proposition 4.1.** *The shift map  $\Phi_A : \widehat{\mathfrak{g}}^* \rightarrow \mathfrak{g}^* \times \mathfrak{a}^*$ , given by  $\Phi_A(\mu, \sigma) = (\mu - \mathcal{A}_\sigma, \sigma)$  is the dual of the trivialization mapping (4.21). Moreover, the shift map is a Poisson map, taking the Lie-Poisson structure (4.22) on  $\widehat{\mathfrak{g}}^*$  into the product Lie-Poisson structure on  $\mathfrak{g}^* \times \mathfrak{a}^*$ .*

**Hamiltonian vector fields.** The Hamiltonian vector field of a function  $H$  on  $\widehat{\mathfrak{g}}^*$  is defined by the equations

$$\dot{\mu} = \mp \left( \text{ad}_{\frac{\delta H}{\delta \mu}}^* \mu + \sigma \circ C \left( \frac{\delta H}{\delta \mu}, \cdot \right) \right), \quad \dot{\sigma} = 0.$$

Notice that the components of  $\sigma$  are Casimirs of the bracket and as such they are constants of motion. In our case these constants of motion will be (a multiple of) the circulation  $\kappa$  and the constants  $\alpha, \beta$ .

## 4.2 The oscillator group

We start by defining the real valued SE(2)-two-cocycle  $B_1 : \text{SE}(2) \times \text{SE}(2) \rightarrow \mathbb{R}$  by

$$B_1((R_\theta, \mathbf{x}), (R_{\theta'}, \mathbf{x}')) = \frac{1}{2} \mathbf{x} \cdot \mathbb{J} R_\theta \mathbf{x}', \quad (4.24)$$

where  $\mathbb{J}$  is the  $2 \times 2$  symplectic matrix

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This cocycle differs from the one used in [35] by a multiplicative factor of  $\frac{1}{2}$ . On the Lie algebra level, we have by (4.20) that the infinitesimal cocycle  $C_1 : \mathfrak{se}(2) \times \mathfrak{se}(2) \rightarrow \mathbb{R}$  is given by

$$C_1((\omega, v_1, v_2), (\omega', v'_1, v'_2)) = v_1 v'_2 - v_2 v'_1.$$

The central extension of SE(2) by  $\mathbb{R}$  using the cocycle  $B_1$  is referred to as the *oscillator group* [33], and will be denoted as  $\text{Osc}$ . The Lie algebra  $\mathfrak{osc}$  of the oscillator group is isomorphic to  $\mathfrak{se}(2) \times \mathbb{R}$ , with Lie bracket

$$\begin{aligned} [(\omega, v_1, v_2; z), (\omega', v'_1, v'_2; z')]_{\mathfrak{osc}} &= \left( [(\omega, v_1, v_2), (\omega', v'_1, v'_2)]_{\mathfrak{se}(2)} ; C_1((\omega, v_1, v_2), (\omega', v'_1, v'_2)) \right) \\ &= (0, v_2 \omega' - \omega v'_2, \omega v'_1 - v_1 \omega' ; v_1 v'_2 - v_2 v'_1). \end{aligned}$$

On the dual  $\mathfrak{osc}^* \cong \mathfrak{se}(2)^* \times \mathbb{R}^*$ , the (minus) Lie-Poisson bracket is given by (4.22), or explicitly by

$$\{F, K\}_{\mathfrak{osc}^*}^-(\mu, \sigma_0) = \{F, K\}_{\mathfrak{se}(2)^*}^-(\mu) - \sigma_0 C_1 \left( \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right),$$

for  $(\mu, \sigma_0) \in \mathfrak{se}(2)^* \times \mathbb{R}^*$ . It is easy to see that this bracket is precisely the bracket (3.18), where  $\sigma_0$  plays the role of the circulation  $\rho\kappa$ . We conclude that the effect of non-zero circulation can be described in terms of the geometry of the oscillator group, and more precisely the cocycle (4.24).

In terms of Lie algebra cohomology, it is easy to show that  $C_1$  is a closed two-cocycle which is not exact. Furthermore, as  $H^2(\mathfrak{se}(2), \mathbb{R})$  is isomorphic to  $\mathbb{R}$ , we have that  $C_1$  determines a generator of the second cohomology. Since isomorphism classes of central extensions are classified by the second Lie algebra cohomology, we will refer to the oscillator group as *the* central extension of SE(2) by  $\mathbb{R}$ .



### 4.3 A central extension of $\text{SE}(2)$ by $\mathbb{R}^3$ .

We now describe the effect of nonzero  $\alpha$  and  $\beta$  in the Poisson bracket (3.17). To this end, we observe that the equations of motion (3.15) for nonzero  $\alpha$  and  $\beta$  can be obtained from the equations where  $\alpha = \beta = 0$  by making the substitution, or *momentum shift*,

$$p_1 \rightsquigarrow p_1 + \sigma_1, \quad p_2 \rightsquigarrow p_2 + \sigma_2, \quad (4.25)$$

where  $\sigma_1 = -\rho\beta$  and  $\sigma_2 = \rho\alpha$ , while the angular momentum  $k$  remains invariant. This momentum shift can be described in geometric terms as follows. Let  $\mathcal{A} : \mathfrak{se}(2) \rightarrow \mathbb{R}^3$  be the linear map  $\mathcal{A}(\omega, v_1, v_2) = (0, v_1, v_2)$ . This map is a one-cocycle on  $\mathfrak{se}(2)$  with values in  $\mathbb{R}^3$  and its derivative is given by

$$\delta\mathcal{A}((\omega, v_1, v_2), (\omega', v'_1, v'_2)) = -\mathcal{A}([(\omega, v_1, v_2), (\omega', v'_1, v'_2)]_{\mathfrak{se}(2)}) = (0, \omega v'_2 - v_2 \omega', v_1 \omega' - \omega v'_1).$$

Now, consider fixed values  $\sigma_1, \sigma_2 \in \mathbb{R}$  and define the linear map  $\mathcal{A}_{(\sigma_1, \sigma_2)} : \mathfrak{se}(2) \rightarrow \mathbb{R}^3$  given by

$$\mathcal{A}_{(\sigma_1, \sigma_2)}(\omega, v_1, v_2) := \langle (0, \sigma_1, \sigma_2), \mathcal{A}(\omega, v_1, v_2) \rangle = \sigma_1 v_1 + \sigma_2 v_2.$$

As  $\mathcal{A}_{(\sigma_1, \sigma_2)}$  is an element of  $\mathfrak{se}(2)^*$ , we may write the momentum shift (4.25) more formally as the map  $\Phi_{\mathcal{A}} : \mathfrak{se}(2)^* \rightarrow \mathfrak{se}(2)^*$  given by

$$\Phi_{\mathcal{A}}(\mu) = \mu - \mathcal{A}_{(\sigma_1, \sigma_2)}, \quad (4.26)$$

for all  $\mu \in \mathfrak{se}(2)^*$ . The minus sign is due to the fact that this is the *active* version of the transformation (4.25): if we denote the new momenta of the system by  $\bar{p}_1, \bar{p}_2$ , then the effect of performing the substitution (4.25) is that the old and the new momenta are related by  $p_1 = \bar{p}_1 + \sigma_1, p_2 = \bar{p}_2 + \sigma_2$ , which is just (4.26).

As the dual of the Lie algebra of the oscillator group is just the Cartesian product  $\mathfrak{se}(2)^* \times \mathbb{R}^*$ , the map  $\Phi_{\mathcal{A}}$  gives rise to a map  $(\mu, \sigma_0) \mapsto (\Phi_{\mathcal{A}}(\mu), \sigma_0)$  on  $\mathfrak{osc}^*$  which we denote by  $\Phi_{\mathcal{A}}$  as well. We now investigate the behavior of the Poisson structure (3.18) under the map  $\Phi_{\mathcal{A}}$ . Since  $\Phi_{\mathcal{A}}$  is a constant shift map, we have for arbitrary functions  $F, K$  on  $\mathfrak{osc}^*$  that

$$\frac{\delta(F \circ \Phi_{\mathcal{A}}^{-1})}{\delta\mu} = \frac{\delta F}{\delta\mu},$$

and similarly for  $K$ , so that  $\{F \circ \Phi_{\mathcal{A}}^{-1}, K \circ \Phi_{\mathcal{A}}^{-1}\}_{\mathfrak{osc}^*}(\mu, \sigma) = \{F, K\}_{\mathfrak{osc}^*}(\mu, \sigma)$ . On the other hand,

$$\begin{aligned} \{F, K\}_{\mathfrak{osc}^*}(\Phi_{\mathcal{A}}^{-1}(\mu), \sigma) &= \{F, K\}_{\mathfrak{se}(2)^*}(\Phi_{\mathcal{A}}^{-1}(\mu)) - \sigma_0 C_1 \left( \frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu} \right) \\ &= \{F, K\}_{\mathfrak{se}(2)^*}(\mu) - \left\langle \mathcal{A}_{(\sigma_1, \sigma_2)}, \left[ \frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu} \right] \right\rangle - \sigma_0 C_1 \left( \frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu} \right) \\ &= \{F, K\}_{\mathfrak{se}(2)^*}(\mu) - (\sigma_0 C_1 - \delta\mathcal{A}_{(\sigma_1, \sigma_2)}) \left( \frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu} \right), \end{aligned} \quad (4.27)$$

so that  $\Phi_{\mathcal{A}}$  takes the Lie-Poisson bracket on  $\mathfrak{osc}^*$  into the bracket (4.27) modified by a cocycle. The cocycle  $\sigma_0 C_1 - \delta\mathcal{A}_{(\sigma_1, \sigma_2)}$  can be made more explicit by noting that

$$(\sigma_0 C_1 - \delta\mathcal{A}_{(\sigma_1, \sigma_2)})((\omega, v_1, v_2), (\omega', v'_1, v'_2)) = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 v'_2 - v_2 v'_1 \\ v_2 \omega' - \omega v'_2 \\ \omega v'_1 - v_1 \omega' \end{bmatrix} = \boldsymbol{\sigma} \cdot \left( \begin{bmatrix} \omega \\ v_1 \\ v_2 \end{bmatrix} \times \begin{bmatrix} \omega' \\ v'_1 \\ v'_2 \end{bmatrix} \right), \quad (4.28)$$

where  $\sigma := (\sigma_0, \sigma_1, \sigma_2)$ . Note that this is precisely the cocycle in the Poisson bracket (3.17) for the specific value of  $\sigma := (\rho\kappa, -\rho\beta, \rho\alpha)$ . We conclude that the effect of introducing nonzero  $\sigma_1, \sigma_2$  is to modify the Lie-Poisson bracket on  $\mathfrak{osc}^*$  by a trivial  $\mathbb{R}^2$ -valued cocycle  $-\delta\mathcal{A}_{(\sigma_1, \sigma_2)}$ . We now let  $C_2$  be equal to  $-\delta\mathcal{A}$ : this is an  $\mathbb{R}^3$ -valued cocycle on  $\mathfrak{se}(2)$ , which can be integrated to a group two-cocycle  $B_2$  on  $\text{SE}(2)$ , given by

$$B_2((R_\theta, \mathbf{x}), (R_{\theta'}, \mathbf{x}')) = (0, R_\theta \mathbf{x}' - \mathbf{x}'). \quad (4.29)$$

As we showed previously, we can describe the cocycle bracket (4.27) by considering the central extension of the oscillator group by the cocycle  $B_2$  given in (4.29). This is equivalent to extending  $\text{SE}(2)$  by  $\mathbb{R}^3$  using the combined cocycle  $B := (B_1, B_2) : \text{SE}(2) \times \text{SE}(2) \rightarrow \mathbb{R}^3$ ; the result is an extension  $\hat{G}$  which is isomorphic to  $\text{SE}(2) \times \mathbb{R}^3$  with multiplication

$$(R_\theta, \mathbf{x}; \mathbf{w}) \cdot (R_{\theta'}, \mathbf{x}'; \mathbf{w}') = (R_{\theta+\theta'}, \mathbf{x} + R_\theta \mathbf{x}'; \mathbf{w} + \mathbf{w}' + B((R_\theta, \mathbf{x}), (R_{\theta'}, \mathbf{x}'))).$$

The infinitesimal cocycle  $C : \mathfrak{se}(2) \times \mathfrak{se}(2) \rightarrow \mathbb{R}^3$  associated to  $B$  is given by  $C = (C_1, C_2)$ , or

$$\begin{aligned} C((\omega, v_1, v_2), (\omega', v'_1, v'_2)) &= (v_1 v'_2 - v_2 v'_1, v_2 \omega' - \omega v'_2, \omega v'_1 - v_1 \omega') \\ &= (\omega, v_1, v_2) \times (\omega', v'_1, v'_2), \end{aligned} \quad (4.30)$$

which is precisely the cocycle appearing on the right-hand side of (4.28). The bracket on the algebra  $\hat{\mathfrak{g}} := \text{Lie}(\hat{G})$  is then given by

$$\begin{aligned} [(\omega, v_1, v_2; \mathbf{z}), (\omega', v'_1, v'_2; \mathbf{z}')]_{\hat{\mathfrak{g}}} &= \left( [(\omega, v_1, v_2), (\omega', v'_1, v'_2)]_{\mathfrak{se}(2)} ; C((\omega, v_1, v_2), (\omega', v'_1, v'_2)) \right) \\ &= (0, v_2 \omega' - \omega v'_2, \omega v'_1 - v_1 \omega'; v_1 v'_2 - v_2 v'_1, v_2 \omega' - \omega v'_2, \omega v'_1 - v_1 \omega'). \end{aligned}$$

**The Lie-Poisson bracket on  $\hat{\mathfrak{g}}^*$ .** We write explicitly the (minus) Lie-Poisson bracket on the dual Lie algebra  $\hat{\mathfrak{g}}^*$ . Note first that as a vector space  $\hat{\mathfrak{g}}^*$  is just  $\mathfrak{se}(2)^* \times \mathbb{R}^3$ , so that an element  $\nu$  of  $\hat{\mathfrak{g}}^*$  can be written as  $\nu := (\mu, \sigma)$  where  $\mu \in \mathfrak{se}(2)^*$  and  $\sigma \in \mathbb{R}^3$ . In view of (4.22) and (4.30) we obtain:

**Proposition 4.2.** *The (minus) Lie-Poisson bracket of functions  $F, K \in C^\infty(\hat{\mathfrak{g}}^*)$  is given by*

$$\{F, K\}_{\hat{\mathfrak{g}}^*}^-(\mu, \sigma) = \{F, K\}_{\mathfrak{se}(2)^*}^-(\mu) - \sigma \cdot \left( \frac{\delta F}{\delta \mu} \times \frac{\delta K}{\delta \mu} \right),$$

where we identify  $\mathfrak{se}(2)$  with  $\mathbb{R}^3$  in the usual way to make sense of the vector product of the functional derivatives.

In coordinates, the Poisson structure of the above proposition is

$$\{F, K\}_{\hat{\mathfrak{g}}^*}^-(\mu, \sigma) = (\nabla_{(\mu, \sigma)} F)^T \begin{pmatrix} 0 & -p_2 - \sigma_2 & p_1 + \sigma_1 & 0 & 0 & 0 \\ p_2 + \sigma_2 & 0 & -\sigma_0 & 0 & 0 & 0 \\ -p_1 - \sigma_1 & \sigma_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} (\nabla_{(\mu, \sigma)} K),$$

where as usual  $\mu = (k, p_1, p_2) \in \mathfrak{se}(2)^*$  and  $\sigma = (\sigma_0, \sigma_1, \sigma_2) \in \mathbb{R}^3$ .

**The equations of motion.** The reduced kinetic energy Hamiltonian  $H(\mu) = \frac{1}{2}\mu \cdot \mathbb{I}^{-1}\mu$  on  $\mathfrak{se}(2)^*$  defined previously in (3.13) gives rise to a Hamiltonian  $H_{\widehat{\mathfrak{g}}^*}$  on  $\widehat{\mathfrak{g}}^*$  as follows:

$$H_{\widehat{\mathfrak{g}}^*}(\mu, \sigma) = H(\mu) + \frac{1}{2}\|\sigma\|^2. \quad (4.31)$$

The equations of motion for this Hamiltonian and the bracket  $\{\cdot, \cdot\}_{\widehat{\mathfrak{g}}^*}^-$  are then given in coordinates by

$$\begin{aligned} \dot{k} &= -p_2v_1 + p_1v_2 - \sigma_2v_1 + \sigma_1v_2, \\ \dot{p}_1 &= p_2\omega + \sigma_2\omega - \sigma_0v_2, \\ \dot{p}_2 &= -p_1\omega - \sigma_1\omega + \sigma_0v_1, \\ \dot{\sigma} &= 0, \end{aligned}$$

that coincide with (3.15) if we put  $\sigma_0 = \rho\kappa$ ,  $\sigma_1 = -\rho\beta$ ,  $\sigma_2 = \rho\alpha$ . We mention that, in addition to the energy  $H$ , these equations possess the conserved quantity

$$\bar{F}(\mu, \sigma) = p_1^2 + p_2^2 + 2\sigma_0k + 2\sigma_1p_1 + 2\sigma_2p_2,$$

which is a Casimir function of the bracket  $\{\cdot, \cdot\}_{\widehat{\mathfrak{g}}^*}^-$ . In fact, for a constant value of  $\sigma$ , the regular level sets of  $\bar{F}$  are symplectic leaves of the symplectic foliation of  $\widehat{\mathfrak{g}}^*$ . It is easily seen that these are paraboloids if  $\sigma_0 \neq 0$ , and cylinders otherwise. The trajectories of the system are defined by the intersection of the level sets of  $\bar{F}$  and  $H$ .

Notice that the Hamiltonian (4.31) is the (reduced) Legendre transform of the left  $\widehat{G}$ -invariant kinetic energy Lagrangian  $\mathcal{L}_{\widehat{G}} : T\widehat{G} \rightarrow \mathbb{R}$  whose value at the identity is  $L_{\widehat{\mathfrak{g}}} : \widehat{\mathfrak{g}} \rightarrow \mathbb{R}$  given by

$$L_{\widehat{\mathfrak{g}}}(\xi, \mathbf{a}) = \frac{1}{2}\xi^T \mathbb{I} \xi + \frac{1}{2}\|\mathbf{a}\|^2, \quad \text{for } (\xi, \mathbf{a}) \in \mathfrak{se}(2) \times \mathbb{R}^3 = \widehat{\mathfrak{g}}.$$

Our discussion can be summarized in the following theorem.

**Theorem 4.3.** *The Chaplygin–Lamb equations (3.15) are of Lie–Poisson type on the dual Lie algebra  $\widehat{\mathfrak{g}}^*$  of the central extension  $\widehat{G}$  of  $\text{SE}(2)$  by  $\mathbb{R}^3$  defined above, and with respect to the left invariant, kinetic energy Hamiltonian determined by (4.31). In particular, the bracket (3.17) coincides with the restriction of the Lie–Poisson bracket on  $\widehat{\mathfrak{g}}^*$  to the Casimir level set defined by  $\sigma_0 = \rho\kappa$ ,  $\sigma_1 = -\rho\beta$ ,  $\sigma_2 = \rho\alpha$ .*

## 5 Nonholonomic LL systems on central extensions

We have seen that the Chaplygin–Lamb equations for the motion of a rigid body in a potential fluid in the presence of circulation are Lie–Poisson equations on  $\widehat{\mathfrak{g}}^*$  corresponding to a pure kinetic energy *left* invariant Lagrangian. Our goal in this section is to add a *left* invariant nonholonomic constraint to the system. The resulting reduced equations, which are consistent with Lagrange–D’Alembert’s principle stating that the constraint force performs no work during the motion, are the so-called *Euler–Poincaré–Suslov (EPS) equations*.

## 5.1 The case of a general central extension

To the best of our knowledge, the derivation of the EPS equations has never been made explicit in the case where the underlying Lie group is a central extension so we take the opportunity to develop the general theory in detail. In section 5.2, we apply this theory to the hydrodynamic Chaplygin sleigh.

**Euler-Poincaré-Suslov equations on a Lie group  $G$ .** In general, a nonholonomic system on a Lie group  $G$  with a left invariant kinetic energy Lagrangian and left invariant constraints is termed an *LL system*. Due to invariance, the dynamics reduce to the Lie algebra  $\mathfrak{g}$ , or to its dual  $\mathfrak{g}^*$  if working with the momentum formulation. We start from a reduced Lagrangian  $L : \mathfrak{g} \rightarrow \mathbb{R}$ , which defines an inertia operator  $\mathbb{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$  by the relation

$$L(\xi) = \frac{1}{2} \langle \mathbb{I}\xi, \xi \rangle, \quad \text{for } \xi \in \mathfrak{g},$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing. The reduced Hamiltonian,  $H : \mathfrak{g}^* \rightarrow \mathbb{R}$ , is then given by

$$H(\mu) = \frac{1}{2} \langle \mu, \mathbb{I}^{-1}\mu \rangle, \quad \text{for } \mu \in \mathfrak{g}^*.$$

The nonholonomic constraints are expressed in terms of  $n$  linearly independent fixed covectors  $\nu_i \in \mathfrak{g}^*$ ,  $i = 1, \dots, n$ : we say that an instantaneous velocity  $\xi \in \mathfrak{g}$  satisfies the constraints if

$$\langle \nu_i, \xi \rangle = 0, \quad i = 1, \dots, n. \quad (5.32)$$

We let  $\mathcal{D} \subset \mathfrak{g}$  be the vector subspace of all velocities satisfying the constraints, and we say that the constraints are nonholonomic if  $\mathcal{D}$  is not a Lie subalgebra of  $\mathfrak{g}$ .

The reduced EPS equations on  $\mathfrak{g}^*$  are given by, see e.g. [3],

$$\dot{\mu} = \text{ad}_{\mathbb{I}^{-1}\mu}^* \mu + \sum_{i=1}^n \lambda_i \nu_i, \quad (5.33)$$

where the multipliers  $\lambda_i$ ,  $i = 1, \dots, n$ , are certain scalars that are uniquely determined by the condition that the constraints (5.32) are satisfied. Explicitly, the Lagrange multipliers are given by

$$\lambda_j = - \sum_{i=1}^n (D^{-1})_{ij} \langle \text{ad}_{\mathbb{I}^{-1}\mu}^* \mu, \mathbb{I}^{-1}\nu_i \rangle, \quad (5.34)$$

where  $D$  is the matrix with components  $D_{ij} := \langle \nu_i, \mathbb{I}^{-1}\nu_j \rangle$ ,  $i, j = 1, \dots, n$ , and  $D^{-1}$  is its inverse.

**Euler-Poincaré-Suslov equations on central extensions.** Now suppose that  $\widehat{G}$  is a central extension of  $G$  by the abelian Lie group  $A$  as explained in section 4. As before, we assume that  $\widehat{G} \cong G \times A$  with the multiplication (4.19). We let  $L$  be a left-invariant Lagrangian on  $G$ , with associated Hamiltonian  $H$ , and we let  $\nu_i \in \mathfrak{g}^*$ ,  $i = 1, \dots, n$  be a set of linearly independent constraint covectors. We now wish to “lift” these data to the central extension  $\widehat{G}$ , so that we can derive the corresponding EPS equations on the co-algebra  $\widehat{\mathfrak{g}}^*$ .

By left translating the co-vectors  $\nu_i \in \mathfrak{g}^*$  one can define left-invariant constraint one-forms  $\epsilon_i$ , on  $G$ , given by  $\epsilon_i(g) = T_g^* L_{g^{-1}} \nu_i \in T_g^* G$ ,  $i = 1, \dots, n$ . Since  $\widehat{G} \cong G \times A$ , these constraint one-forms naturally induce constraint one-forms  $\widehat{\epsilon}_i$  on  $\widehat{G}$ , given by  $\widehat{\epsilon}_i(g, \alpha) = (\epsilon_i(g), 0)$ . Likewise, the co-vectors  $\nu_i \in \mathfrak{g}^*$  can be lifted to co-vectors  $\widehat{\nu}_i = (\nu_i, 0)$  in  $\widehat{\mathfrak{g}}^*$ , and we have that  $\widehat{\epsilon}_i(g, \alpha) = T_{(g, \alpha)}^* L_{(g, \alpha)^{-1}} \widehat{\nu}_i$ .

Secondly, we define the left invariant, kinetic energy Hamiltonian  $H_{T^*\widehat{G}} : T^*\widehat{G} \rightarrow \mathbb{R}$ , whose value at the identity is given by

$$H_{\widehat{\mathfrak{g}}^*}(\mu, \sigma) = H(\mu) + \frac{1}{2} \|\sigma\|^2, \quad \text{for } (\mu, \sigma) \in \mathfrak{g}^* \times \mathfrak{a}^* = \widehat{\mathfrak{g}}^*, \quad (5.35)$$

where  $\|\cdot\|^2$  denotes any positive definite, quadratic form on  $\mathfrak{a}^*$ . For convenience we write

$$H_{\widehat{\mathfrak{g}}^*}(\mu, \sigma) = \left\langle (\mu, \sigma), \widehat{\mathbb{I}}^{-1}(\mu, \sigma) \right\rangle$$

where the non-degenerate, extended, inertia tensor  $\widehat{\mathbb{I}} : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}^*$  is determined from (5.35).

We have thus extended the left invariant constraints and kinetic energy on  $G$  to define an LL system on the central extension  $\widehat{G}$ . The corresponding EPS equations (5.33) become,

$$\begin{aligned} (\dot{\mu}, \dot{\sigma}) &= \widehat{\text{ad}}_{\widehat{\mathbb{I}}^{-1}(\mu, \sigma)}^*(\mu, \sigma) + \sum_{i=1}^n \lambda_i \widehat{\nu}_i \\ &= (\text{ad}_{\mathbb{I}^{-1}\mu}^* \mu + \sigma \circ C(\mathbb{I}^{-1}\mu, \cdot), 0) + \sum_{i=1}^n \lambda_i (\nu_i, 0), \end{aligned}$$

where  $\widehat{\text{ad}}^*$  (respectively,  $\text{ad}^*$ ) denotes the infinitesimal coadjoint action on  $\widehat{\mathfrak{g}}^*$  (respectively, on  $\mathfrak{g}^*$ ). Here, the Lie algebra two-cocycle  $C : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{a}$  is defined by (4.20).

The above equations can be written as

$$\dot{\mu} = \text{ad}_{\mathbb{I}^{-1}\mu}^* \mu + \sigma \circ C(\mathbb{I}^{-1}\mu, \cdot) + \sum_{i=1}^n \lambda_i \nu_i, \quad \dot{\sigma} = 0, \quad (5.36)$$

and one finds the following expression for the multipliers:

$$\lambda_j = - \sum_{i=1}^n (D^{-1})_{ij} \left( \langle \text{ad}_{\mathbb{I}^{-1}\mu}^* \mu, \mathbb{I}^{-1}\nu_i \rangle + \langle \sigma, C(\mathbb{I}^{-1}\mu, \mathbb{I}^{-1}\nu_i) \rangle \right). \quad (5.37)$$

Notice that these equations reduce to (5.33), (5.34) if  $\sigma = 0$ .

**Existence of invariant measures.** It is natural to ask whether the equations (5.36) and (5.37) possess an invariant measure. We settle this question using the criterion of Jovanović [13] for the existence of an invariant measure for the EPS equations on the Lie algebra of a non-compact group. We assume that there is only one constraint, so that  $n = 1$ ; the case of multiple constraints can be

dealt with in a similar way. Following [13], the necessary and sufficient condition for equations (5.32), (5.33) to have an invariant measure is that the constraint covector  $\widehat{\nu} = \widehat{\nu}_1 \in \widehat{\mathfrak{g}}^*$  satisfies

$$\frac{1}{\langle \widehat{\nu}, \mathbb{I}^{-1} \widehat{\nu} \rangle} \widehat{\text{ad}}_{\mathbb{I}^{-1} \widehat{\nu}}^* \widehat{\nu} + \widehat{T} = c \widehat{\nu}, \quad c \in \mathbb{R}, \quad (5.38)$$

where  $\widehat{T} \in \widehat{\mathfrak{g}}^*$  is defined by the relation  $\langle \widehat{T}, \widehat{\xi} \rangle = \text{trace}(\widehat{\text{ad}}_{\widehat{\xi}})$ ,  $\widehat{\xi} \in \widehat{\mathfrak{g}}$ .

However, for  $(\xi, a), (\eta, b) \in \mathfrak{g} \times \mathfrak{a} = \widehat{\mathfrak{g}}$  we have

$$\widehat{\text{ad}}_{(\xi, a)}(\eta, b) = (\text{ad}_{\xi} \eta, C(\xi, \eta)).$$

It follows that the operator  $\widehat{\text{ad}}_{(\xi, a)} : \widehat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{a} \rightarrow \widehat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{a}$  has matrix representation

$$\begin{pmatrix} \text{ad}_{\xi} & 0 \\ C(\xi, \cdot) & 0 \end{pmatrix}.$$

Hence  $\text{trace}(\widehat{\text{ad}}_{(\xi, a)}) = \text{trace}(\text{ad}_{\xi})$  and we can write  $\widehat{T} = (T, 0) \in \mathfrak{g}^* \times \mathfrak{a}^*$ , where  $T \in \mathfrak{g}^*$  is defined by  $\langle T, \xi \rangle = \text{trace}(\text{ad}_{\xi})$  for  $\xi \in \mathfrak{g}$ . In addition, since  $\widehat{\nu} = (\nu, 0)$ , we can write (5.38) in components as

$$\frac{1}{\langle \nu, \mathbb{I}^{-1} \nu \rangle} \text{ad}_{(\mathbb{I}^{-1} \nu, 0)}^* (\nu, 0) + (T, 0) = c(\nu, 0), \quad c \in \mathbb{R}.$$

Therefore, the condition (5.38) is equivalent to

$$\frac{1}{\langle \nu, \mathbb{I}^{-1} \nu \rangle} \text{ad}_{\mathbb{I}^{-1} \nu}^* \nu + T = c \nu, \quad c \in \mathbb{R},$$

which is precisely the necessary and sufficient condition for the equations (5.33), (5.34) to possess an invariant measure. This analysis can be generalized to the case where the number  $n$  of constraints is arbitrary. This shows:

**Theorem 5.1.** *The Euler-Poincaré-Suslov equations (5.36) and (5.37) on the dual Lie algebra  $\widehat{\mathfrak{g}}^*$  of the central extension  $\widehat{G}$  of the Lie group  $G$ , possess an invariant measure for an arbitrary value of  $\sigma \in \mathfrak{a}^*$  if and only if they possess an invariant measure for the specific value  $\sigma = 0$ . In other words, such an invariant measure exists if and only if the Euler-Poincaré-Suslov equations (5.33), (5.34) on  $\mathfrak{g}^*$  possess an invariant measure.*

## 5.2 The hydrodynamic Chaplygin sleigh with circulation

We are now ready to consider the mechanical system of our interest which is the generalization of the hydrodynamic version of the Chaplygin sleigh treated in [10] to the case when there is circulation around the body. Recall that the classical Chaplygin sleigh problem (going back to 1911, [7]) describes the motion of a planar rigid body with a knife edge (a blade) sliding on a horizontal plane. The nonholonomic constraint forbids the motion in the direction perpendicular to the blade. In its hydrodynamic version, the body is surrounded by a potential fluid and the nonholonomic constraint models the effect of a very effective fin or keel, see [10].

With the notation from section 2, we let  $\{\mathbf{E}_1, \mathbf{E}_2\}$  be a body frame located at the contact point of the sleigh and the plane, and so that the  $\mathbf{E}_1$ -axis is aligned with the blade (see Figure 1). The resulting nonholonomic constraint is given by  $v_2 = 0$ , and is clearly left invariant under the action of  $\text{SE}(2)$ , as it is solely written in terms of the velocity of the body as seen in the body frame.

In the absence of constraints, the motion of the body is described by the Chaplygin–Lamb equations (3.15) that, as we have shown (Theorem 4.3), are Lie-Poisson equations on the dual Lie algebra of the central extension  $\widehat{G}$  of  $\text{SE}(2)$  and with respect to a left invariant kinetic energy Hamiltonian. The reduced nonholonomic equations are thus of EPS type on  $\widehat{\mathfrak{g}}^*$ . Note that the co-vector  $(0, 0, 1) \in \widehat{\mathfrak{g}}^*$  annihilates all elements  $\xi = (\omega, v_1, v_2) \in \mathfrak{se}(2)$  that satisfy the constraints. Thus, in agreement with the results of Theorem 4.3, by putting  $\mu = (k, p_1, p_2) \in \mathfrak{se}(2)^*$  and  $\sigma = (\rho\kappa, -\rho\beta, \rho\alpha) \in \mathbb{R}^3$ , we get the following explicit expression for the reduced nonholonomic equations (5.36),

$$\begin{aligned}\dot{k} &= v_2 p_1 - v_1 p_2 - \rho(\alpha v_1 + \beta v_2), \\ \dot{p}_1 &= \omega p_2 - \kappa \rho v_2 + \rho \alpha \omega, \\ \dot{p}_2 &= -\omega p_1 + \kappa \rho v_1 + \rho \beta \omega + \lambda,\end{aligned}\tag{5.39}$$

where the multiplier  $\lambda$  is determined from the condition  $v_2 = 0$ .

**The total inertia tensor and the circulation constants  $\alpha, \beta$ .** The expression for  $\mathbb{I}_{\mathcal{B}}$  with respect to the body frame  $\{\mathbf{E}_1, \mathbf{E}_2\}$  is given by (2.6) where  $m$  is the mass of the body,  $(a, b)$  are body coordinates of the center of mass, and  $\mathcal{I}$  is the moment of inertia of the body about the center of mass. While this simple expression is independent of the body shape, an explicit expression for the tensor of adjoint masses  $\mathbb{I}_{\mathcal{F}}$  can be given explicitly only for rather simple geometries. A simple yet interesting one is that for an elliptical uniform planar body with the semi-axes of length  $A > B > 0$ . Assume that the origin has coordinates  $(r, s)$  with respect to the frame that is aligned with the principal axes of the ellipse and that the coordinate axes  $\mathbf{E}_1, \mathbf{E}_2$  are not aligned with the axes of the ellipse, forming an angle  $v$  (measured counter-clockwise), as illustrated in Figure 1.

For this geometry, starting from the formula (2.11) for the added inertia tensor in a body frame whose axes are aligned with the symmetry axes of the ellipsoid, one can show that

$$\mathbb{I}_{\mathcal{F}} = \rho\pi \begin{pmatrix} \frac{(A^2-B^2)^2}{4} + s^2(B^2 \cos^2 v + A^2 \sin^2 v) & s(B^2 \cos^2 v + A^2 \sin^2 v) & -r(A^2 \cos^2 v + B^2 \sin^2 v) + \frac{1}{2}s(A^2 - B^2) \sin(2v) \\ +r^2(A^2 \cos^2 v + B^2 \sin^2 v) - rs(A^2 - B^2) \sin(2v) & -\frac{1}{2}r(A^2 - B^2) \sin(2v) & +\frac{1}{2}s(A^2 - B^2) \sin(2v) \\ s(B^2 \cos^2 v + A^2 \sin^2 v) & B^2 \cos^2 v + A^2 \sin^2 v & \frac{A^2-B^2}{2} \sin(2v) \\ -\frac{1}{2}r(A^2 - B^2) \sin(2v) & & \\ -r(A^2 \cos^2 v + B^2 \sin^2 v) & \frac{A^2-B^2}{2} \sin(2v) & A^2 \cos^2 v + B^2 \sin^2 v \\ +\frac{1}{2}s(A^2 - B^2) \sin(2v) & & \end{pmatrix}. \tag{5.40}$$

The total inertia tensor,  $\mathbb{I} = \mathbb{I}_{\mathcal{B}} + \mathbb{I}_{\mathcal{F}}$ , of the fluid-body system is then given by

$$\mathbb{I} = \begin{pmatrix} \mathcal{I} + m(a^2 + b^2) & -mb + \rho\pi[s(B^2 \cos^2 v + A^2 \sin^2 v) - \frac{1}{2}r(A^2 - B^2) \sin(2v)] & ma + \rho\pi[-r(A^2 \cos^2 v + B^2 \sin^2 v) + \frac{1}{2}s(A^2 - B^2) \sin(2v)] \\ +\rho\pi[\frac{(A^2-B^2)^2}{4} + s^2(B^2 \cos^2 v + A^2 \sin^2 v) + r^2(A^2 \cos^2 v + B^2 \sin^2 v) - rs(A^2 - B^2) \sin(2v)] & m + \rho\pi[B^2 \cos^2 v + A^2 \sin^2 v] & \rho\pi[\frac{A^2-B^2}{2} \sin(2v)] \\ -mb + \rho\pi[s(B^2 \cos^2 v + A^2 \sin^2 v) - \frac{1}{2}r(A^2 - B^2) \sin(2v)] & & \\ ma + \rho\pi[-r(A^2 \cos^2 v + B^2 \sin^2 v) + \frac{1}{2}s(A^2 - B^2) \sin(2v)] & \rho\pi[\frac{A^2-B^2}{2} \sin(2v)] & m + \rho\pi(A^2 \cos^2 v + B^2 \sin^2 v) \end{pmatrix}.$$

Notice that in the presence of the fluid, if  $v \neq n\frac{\pi}{2}, n \in \mathbb{Z}$ , the coefficient  $\mathbb{I}_{23} = \mathbb{I}_{32}$  is non-zero. This can never be the case if the sleigh is moving in vacuum as one can see from the expression given for  $\mathbb{I}_{\mathcal{B}}$  in (2.6). The appearance of this non-zero term leads to interesting dynamics that, in the absence of circulation were studied in [10]. For the above geometry, the circulation constants  $\alpha$  and  $\beta$  defined by (3.16) are computed to be:

$$\alpha = \kappa(r \cos v + s \sin v), \quad \beta = \kappa(-r \sin v + s \cos v).$$

Notice that, in the presence of circulation, the two constants,  $\alpha$  and  $\beta$ , can vanish simultaneously only if  $r = s = 0$ , that is, only if the contact point is at the center of the ellipse.

In the sequel we assume that the shape of the sleigh is arbitrary convex and that its center of mass does not necessarily coincide with the origin, which leads to the general total inertia tensor

$$\mathbb{I} = \begin{pmatrix} J & -L_2 & L_1 \\ -L_2 & M & Z \\ L_1 & Z & N \end{pmatrix},$$

and with arbitrary circulation constants  $\alpha, \beta$ .

**Detailed equations of motion.** In accordance with (5.34), the multiplier  $\lambda$  is given by:

$$\lambda = -\frac{1}{(\mathbb{I}^{-1})_{33}} \left( \mathbb{I}^{-1} \begin{pmatrix} v_2 p_1 - v_1 p_2 - \rho(\alpha v_1 + \beta v_2) \\ \omega p_2 - \kappa \rho v_2 + \rho \alpha \omega \\ -\omega p_1 + \kappa \rho v_1 + \rho \beta \omega \end{pmatrix} \right)_3,$$

where

$$\mathbb{I}^{-1} = \frac{1}{\det(\mathbb{I})} \begin{pmatrix} MN - Z^2 & ZL_1 + NL_2 & -ZL_2 - ML_1 \\ ZL_1 + NL_2 & JN - L_1^2 & -L_1L_2 - JZ \\ -ZL_2 - ML_1 & -L_1L_2 - JZ & JM - L_2^2 \end{pmatrix}.$$

A long but straightforward calculation shows that, by expressing  $\omega, v_1$  and  $v_2$  in terms of  $k, p_1, p_2$ , substituting into (5.39), and enforcing the constraint  $v_2 = 0$ , one obtains:

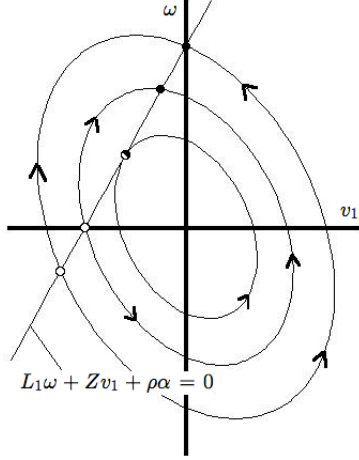
$$\begin{aligned} \dot{\omega} &= \frac{1}{D} (L_1 \omega + Z v_1 + \rho \alpha) (L_2 \omega - M v_1), \\ \dot{v}_1 &= \frac{1}{D} (L_1 \omega + Z v_1 + \rho \alpha) (J \omega - L_2 v_1), \end{aligned} \tag{5.41}$$

where we set  $D = \det(\mathbb{I})(\mathbb{I}^{-1})_{33} = MJ - L_2^2$ . Note that  $D > 0$  since  $\mathbb{I}$  and  $\mathbb{I}^{-1}$  are positive definite. Note as well that if  $\alpha = 0$  we recover the system with zero circulation treated in [10] so from now on we assume  $\alpha \neq 0$ .

The full motion of the sleigh on the plane is determined by the reconstruction equations which, in our case with  $v_2 = 0$ , reduce to

$$\dot{\theta} = \omega, \quad \dot{x} = v_1 \cos \theta, \quad \dot{y} = v_1 \sin \theta.$$





**Figure 5:** Reduced phase portrait. The line  $\ell$  consists of equilibria that are either stable (filled dots) or unstable (empty dots). The trajectories are contained in the level surfaces of the Hamiltonian  $H$  that are ellipses on the  $v_1$   $\omega$  plane.

The reduced energy integral has

$$H = \frac{1}{2} (J\omega^2 + Mv_1^2 - 2L_2\omega v_1),$$

and its level sets are ellipses on the  $(\omega, v_1)$ -plane.

As seen from the equations, the straight line  $\ell = \{L_1\omega + Zv_1 + \rho\alpha = 0\}$  consists of equilibrium points for the system. Each of these equilibria corresponds to a uniform circular motion on the plane along a circumference of radius  $|\frac{v_1}{\omega}|$ .

Notice that if  $Z = L_1 = 0$  the line  $\ell$  of equilibria disappears. In fact, it is shown in [10] that in the absence of circulation the system possesses an invariant measure only for this specific value of the parameters. In view of Theorem 5.1 we conclude

**Proposition 5.2.** *The equations of motion (5.41) possess an invariant measure if and only if  $Z = L_1 = 0$ .*

In this case we obtain simple harmonic motion on the reduced plane  $\omega, v_1$ . The reduced phase space in the case where  $Z \neq 0$  and  $L_1 \neq 0$  is illustrated in Figure 5.

For any value of the parameters the reduced system (5.41) can be checked to be Hamiltonian with respect to the following Poisson bracket of functions of  $\omega, v_1$  that can be obtained using the results of [12]

$$\{F_1, F_2\} := -\frac{1}{D} (L_1\omega + Zv_1 + \rho\alpha) \left( \frac{\partial F_1}{\partial \omega} \frac{\partial F_2}{\partial v_1} - \frac{\partial F_1}{\partial v_1} \frac{\partial F_2}{\partial \omega} \right).$$

The invariant symplectic leaves consist of the semi-planes separated by the equilibria line  $\ell$  and the zero-dimensional leaves formed by the points on this line.

The integration of these equations as well as a detailed study of the motion of the sleigh on the plane will be given in [11].

## Conclusions and further work

In this paper, we presented an interesting example of a nonholonomic hydrodynamical system. We have shown how the Hamiltonian structure of the Chaplygin-Lamb equations can be understood in terms of central extensions of Lie groups, and used this geometric construction to formulate the equations of motion for our problem in EPS form. This structure of the equations was exploited to study the existence of invariant measures for the hydrodynamic Chaplygin sleigh with circulation.

The physical relevance of our model lies in the fact that in the design of underwater vehicles the nonholonomic constraint can be interpreted as a simple model for a fin. From the mathematical point of view, furthermore, our example is remarkable since its equations of motion are explicitly integrable, and describe interesting dynamics as we show in [11].

For the future, we intend to study the motion of the hydrodynamic Chaplygin sleigh coupled to *point vortices* in the fluid [25]. The equations of motion for interacting point vortices and rigid bodies (without nonholonomic constraints) were recently derived in [31, 1] and since then there have been significant efforts towards discerning integrability and chaoticity [27, 29] and towards uncovering the underlying geometry of these models [34]. We plan on coupling the nonholonomic Chaplygin sleigh with one or several point vortices in the flow, taking these models as our starting point. We hope to report with progress on these problems in the near future.

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